Retake Test 2 Computational Methods of Science, February 3, 2023

Duration: 1 hour.
In front of the questions, one finds the points. The sum of the points plus 1 gives the end mark for this test. Criteria used for the grading are: demonstration of understanding, logical reasoning, correct use of terminology, correctness of results.

1. Consider the following partial differential equation on $[0, 1]$:

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial x}(\cos(x)\frac{\partial u}{\partial x})$$

for $t > 0$. The boundary conditions are $u(0, t) = 0$, $u_x(1, t) = 0$. Furthermore, we have the initial conditions $u(x, 0) = \sin(\pi x/2)$ and $u_t(x, 0) = 0$.

   (a) (2.5 points) Give the difference equations and their initial conditions that result when applying a second-order accurate finite-volume discretization in space on an equidistant grid with grid size Δx to the given initial boundary value problem. Apply the left boundary condition at the grid point and the right in the middle between two grid points.

   (b) (1.0 points) Show how the difference equations of part a are written in vector form

$$\frac{d^2}{dt^2}\mathbf{u} = A\mathbf{u} \tag{1}$$

   and define $\mathbf{u}$, and A. Also give the vector form of the initial conditions. Is A symmetric or not.

   (c) (1.5 points) Use the difference/Fourier method to get an estimate of the eigenvalues of A. Since the elements of A depend on the grid points, and hence each row is different, take a typical row and replace the cos by its bound such that the elements of that row are maximized (in absolute value) and the elements no longer depend on the grid points. You may use that $\sin^2\theta = \frac{1}{2} - \frac{1}{2}\cos 2\theta$.

   (d) (1.0 points) Transform the system of second-order differential equations in (1) to a system of first-order differential equations and write it in the form

$$\frac{d}{dt}\mathbf{w} = B\mathbf{w}. \tag{2}$$

   Define $\mathbf{w}$ and B. Don't forget the initial conditions!

   (e) Assume λ is an eigenvalue of the matrix A and that $-\alpha \le \lambda \le 0$ for some $\alpha > 0$.

      i. (1.5 points) What is a good aproximation of α based on the Fourier eigenvalues found in part c. If you were not able to find an estimate of λ in part c, you may use as estimate $-(\sin\theta/\Delta x)^2$ where Δx is the mesh width in x-direction. Argue that the eigenvalues of B are purely imaginary and express them in the eigenvalues of A and give lower and upper bound.

      ii. (0.5 points) Show that for purely imaginary eigenvalues the solution of (2) will not go to zero for $t \to \infty$?

   (f) (1.0 points) Consider for the problem $y' = f(t, y)$ the time-integration method $w_{n+1} = w_{n-1} + 2\Delta t f(t_n, w_n))$ which has an amplification factor $\rho(z)$ with property $|\rho(z)| = 1$ for $z = iy$ with y real and $y \in [-1, 1]$. Express the maximum allowed time step Δt in terms of Δx when we apply this time integration method to the first-order system (2). If you were not able to solve part e, you may assume that the eigenvalues μ of B are on the interval $[-\frac{4i}{(\Delta x)^3}, \frac{4i}{(\Delta x)^3}]$.